Perfectoid towers arising from Frobenius lifts

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This talk is based on [Ish24].

Fix a prime number p. In [INS25], they introduce the notion of *perfectoid towers* as a "tower-theoretic" generalization of perfectoid rings to study mixed characteristic commutative rings. Roughly speaking, a perfectoid tower is a tower of commutative rings such that whose colimit is a perfectoid ring and it gives a good approximation of it. We can also define the *tilt* of a perfectoid tower, which exchanges the characteristic of the rings in the tower from mixed characteristic to positive characteristic.

One of the most fundamental examples of perfectoid towers is

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[p^{1/p}] \hookrightarrow \mathbb{Z}_p[p^{1/p^2}] \hookrightarrow \mathbb{Z}_p[p^{1/p^3}] \hookrightarrow \cdots,$$

where $\mathbb{Z}_p$ is the ring of p-adic integers. Its tilt is

$$\mathbb{F}_p[|T|] \xrightarrow{F} \mathbb{F}_p[|T|] \xrightarrow{F} \mathbb{F}_p[|T|] \xrightarrow{F} \mathbb{F}_p[|T|] \xrightarrow{F} \cdots,$$

where $\mathbb{F}_p[|T|]$ is the formal power series ring over $\mathbb{F}_p$ and F is the Frobenius map.

However, the only known examples of perfectoid towers arising from Noetherian rings were those constructed from *(log-)regular local rings* in [INS25]. In [Ish24], we focus on the notion of *Frobenius lifts* of commutative rings and construct perfectoid towers from a large class of (possibly singular) Noetherian rings including affine semigroup rings and Stanley–Reisner rings.

Theorem (Special case of [Ish24, Theorem 1.2])**.** *Let R be a mixed-characteristic complete Noetherian local ring such that p is a non-zero-divisor on R and R/pR is F-finite and reduced. Assume that there exists a* Frobenius lift $\varphi\colon R \to R$ *of R, i.e., a ring endomorphism φ of R such that its reduction modulo p is the Frobenius map F on R/pR. Then there exists a perfectoid tower*

$$R \hookrightarrow R^{1/p} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p}] \hookrightarrow R^{1/p^2} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^2}] \hookrightarrow R^{1/p^3} \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^3}] \hookrightarrow \cdots,$$

where R^{1/p^i} is a finite colimit $\operatorname{colim}\{R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} R\}$ *of R obtained by i-times iterating φ. Its tilt is*

$$R/pR[|T|] \xrightarrow{F} R/pR[|T|] \xrightarrow{F} R/pR[|T|] \xrightarrow{F} R/pR[|T|] \xrightarrow{F} \cdots.$$

By virtue of this theorem, we can unify the known examples and its computation of tilts. Actually, the existence of a Frobenius lift is somewhat restrictive but still there are many examples of commutative rings admitting Frobenius lifts:

Corollary. *The above theorem ensures the existence of perfectoid towers arising from the following rings:*

- *Affine semigroup rings in mixed characteristic, e.g., rings of the form $\mathbb{Z}_p[|H|]$ for an affine semigroup H with $\varphi(\mathbf{t}^h) := \mathbf{t}^{ph}$.*

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- *Stanley–Reisner rings in mixed characteristic, e.g., rings of the form* $\mathbb{Z}_p[|x_1, \dots, x_n|]/I$ *for an ideal* I *generated by square-free monomials with* $\varphi(x_i) \coloneqq x_i^p$.
- *Take integers* (a, b, c) *such that at most one of them is even. Then* $\mathbb{Z}_2[|X, Y, Z|]/(X^a + Y^b + Z^c, Y^{2b} + Y^b Z^c + Z^{2c})$ *with* $\varphi(X) = X^2, \varphi(Y) = Y^2, \varphi(Z) = Z^2$.
- *Section rings* $\bigoplus_{m \geq 0} H^0(\mathcal{A}, \mathcal{L}^{\otimes m})$ *of the canonical lift* $(\mathcal{A}, \mathcal{L})$ *of* (A, L) *consisting of an ordinary Abelian variety* A *over a perfect field of characteristic* p *and an ample line bundle* L *on* A.

In this talk, we introduce the notion of perfectoid towers and Frobenius lifts of rings and explain the above theorem and examples in detail.

References

[INS25] S. Ishiro, K. Nakazato, and K. Shimomoto, *Perfectoid Towers and Their Tilts : With an Application to the Étale Cohomology Groups of Local Log-Regular Rings*, (2025). `http://arxiv.org/abs/2203.16400`. To appear in Algebra and Number Theory.

[Ish24] R. Ishizuka, *Perfectoid towers generated from prisms*, (2024). `http://arxiv.org/abs/2409.15785`.